

XIV... The Mathematics of the Simple Oscillator

Introduction:

These kind of systems are found nearly everywhere, and in college, many of your students will study these in considerable detail through math and physics. The previous two activities exposed your students to the two important aspects of a system (amplitude and frequency) that can be easily observed. The amplitude changes in a periodic manner that illustrates how sines and cosines can be important in describing how some systems work. Without any damping, the amplitude plots from each cycle can be plotted on top of each other to show how the motion is periodic just like a sine or cosine curve. In this activity, some of the mathematics will be explored to further describe how these systems work. Suitable for advanced students in Honors Math or Physics.

1) The oscillation of the magnet can be represented by the following differential equation:

$$m \frac{d^2 X}{dt^2} = -k X^2$$

where **m** is the mass of the magnet and card, **X** is the displacement of the light spot, and **k** is the coefficient that represents the springiness of the string (Hook's Constant)

2) If the magnet is not pushed as it begins its swing, a solution to this equation is given by:

$$X(t) = X(0) e^{-if t}$$

where $X(0)$ is the amplitude of the swing, and f is the natural oscillation frequency and i is the square root of -1.

3) Substitute solution into the differential equation and determine how the quantities $X(0)$ and f are related to the parameters that define this particular system, namely, m and k , as follows:

$$m \frac{d^2 [X(0) e^{-if t}]}{dt^2} = -k X(0) e^{-if t}$$

$$m X(0) f^2 e^{-if t} = -k X(0) e^{-if t}$$

4) Note the factors that cancel on both sides, leaving the natural frequency now defined by m and k which are properties of the magnetometer:

$$f = \left[\frac{k}{m} \right]^{\frac{1}{2}}$$

Note: This is actually a measure of the angular frequency in radians/sec. To get the frequency in cycles/sec divide 'f' by 2π

Plotting Activity:

5) The solution we have found can be translated into the familiar sines and cosines by using the

$$e^{-ift} = \cos(ft) + i \sin(ft)$$

If you start tracking the oscillation when the spot is at its maximum displacement at time $t=0$, then plot the formula

$$X(t) = X(0) \cos(ft)$$

If you start the measurement when the displacement is at its null 'minimum' position, then use the formula:

$$X(t) = X(0) \sin(ft)$$

6) Students will plot $X(t)$ for the appropriate case using the measured frequency of oscillation, and compare the plotted sin/cosine curve against what they observed for the magnetometer. They will note that although they were able to correctly predict when the amplitude would reach zero at each swing, they will not be able to reproduce the declining amplitude values. That is because we have not allowed for damping in the above physical model.

Including Damping in our Physical Model

7) A damped oscillator operates in the presence of a retarding force, in this case air friction, which physically occurs because of the speed of the movement through the resisting medium. The higher the speed, the more resistance. The lower the speed, the less the resistance. The differential equation, in this case, has a second term on the right-hand side that is proportional to the speed of the motion (what we call in this equation, its first derivative in time). The equation would look like this:

$$m \frac{d^2 X}{dt^2} = -k X + R \left[\frac{dX}{dt} \right]$$

8) The first term (-kX) is the restorative tension force in the string, and the damping force has an opposite sign to this. Moreover, air friction can be measured to be proportional to the square of the speed of the object. The quantity, R, is a coefficient of friction that depends on the viscosity of the medium and the geometry (cross sectional area) of the body moving through the medium

9) The solution to this equation is a little more complicated than for the previous undamped system because the amplitude X(0) now depends on time as well. The damping introduces an exponential decay factor:

$$e^{-Dt}$$

10) The formal solution to the above differential equation is probably beyond even your most able student to work out, but we can approximate the solution by ‘guessing’ that it might look like this:

$$X(t) = X(0) e^{-Dt} e^{-ift}$$

11) This solution has the desired property that it is a periodic function (the second exponential factor) and that it decays in time (the first exponential factor). The quantity ‘D’ in the exponent determines how rapidly the amplitude will decrease after each swing. A small value, indicating little friction, causes the amplitudes to barely change. A large factor, representing large friction, causes a rapid decrease in the swing amplitude. To estimate a value for D, students can plot the amplitudes of each swing, and then plot the factor e^{-Dt} to find a value for D that matches the plot. By experimenting with different types of thread or wire suspending the magnet in the bottle, a variety of different values for D will result.